Universality classes and the Anderson transition in two dimensions

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Abstract

The Anderson transition is a disorder driven quantum phase transition between metallic and insulating phases. In contrast to the common belief that two dimensional (2D) systems are always insulating and that the Anderson transition does not occur in 2D, in certain universality classes 2D systems can be metallic. We review the recent development of the theory of the Anderson transition in 2D. There are ten universality classes: three Wigner-Dyson classes, three chiral universality classes, and four Bogoliubov-de Gennes classes. We report results for critical exponents and distributions of conductance for the symplectic universality class. We emphasize that, on the one hand, the existence of a topological insulating phase does not alter the value of the critical exponent, while on the other, it strongly affects the form of the conductance distribution at the transition.

INTRODUCTION

In systems with a periodic potential, the wave functions of electrons are extended consistent with Bloch's theorem. In strongly disordered systems, however, due to destructive quantum interference, the envelope of the electron wave functions decays exponentially on a length scale called the localization length ξ,

$$\psi(\vec{r}) = a(\vec{r} - \vec{r}_0) \exp(-|\vec{r} - \vec{r}_0|/\xi) \,. \tag{1}$$

Here, $\vec{r}_0$ is the localization center and a is a random function with a decay that is weaker than exponential. As a function of disorder, a localization-delocalization transition, called the Anderson transition [1], occurs. This transition is characterized by the divergence of the localization length ξ,

$$\xi \sim \frac{1}{|x - x_{\mathrm{c}}|^{\nu}} \,, \tag{2}$$

where x is a parameter such as Fermi energy E or the strength of the random potential W that is used to drive the transition, and ν is the critical exponent. (In the metallic phase, ξ is again finite but is there interpreted as a correlation length.) The value of the exponent ν is thought to be highly universal depending only on the universality class and the dimension (1D, 2D, 3D etc.) of the system.

The transition has been widely studied analytically [2], numerically [3], and experimentally [4], not only in semiconductors, but also in optical [5] and acoustic systems [6]. The recent development of experimental techniques that utilize Bose-Einstein condensation has shed new light on the Anderson transition [7]. In this report, we describe the classification into universality classes, and report recent numerical results for one of the universality classes, the symplectic universality class, as an example.

UNIVERSALITY CLASSES

Random Hamiltonian matrices are classified according to whether the system is invariant under the operations of time reversal T and spin rotation S. Systems with both time reversal and spin rotation symmetries comprise the orthogonal class, systems with time reversal symmetry but with broken spin rotation symmetry comprise the symplectic class, and systems with broken time reversal symmetry comprise the unitary class. (Once

T	S	Symmetry class
Yes	Yes	Orthogonal
Yes	No	Symplectic
No	not relevant	Unitary

TABLE I: Wigner-Dyson classes and symmetry

time reversal symmetry is broken, spin rotation symmetry is no longer relevant). These are called Wigner-Dyson classes [8, 9]. See Table I.

Recently, it has been found necessary to extend the Wigner-Dyson classification [10–12] to describe a wider variety of random systems such as disordered superconductors. The new classification is based on Lie algebra.

Let H be an $N \times N$ Hermitian matrix and let $X = iH$. Then, X is anti-Hermitian. Such matrices X are elements of the Lie algebra $\mathrm{u}(N)$, and $\exp(X)$ elements of the Lie group $\mathrm{U}(N)$. This is the unitary class in the Wigner-Dyson classification. In the absence of time reversal symmetry (or any other special symmetries) the Hamiltonian of a disordered system is in this class.

Any Hermitian matrix can be decomposed as $H = H_1 + iH_2$, where H_1 is a real symmetric matrix and H_2 a real antisymmetric matrix. The matrices H_2 are the elements of a Lie algebra that is a subalgebra of $\mathrm{u}(N)$. The corresponding Lie group is $\mathrm{SO}(N)$, which is a subgroup of $\mathrm{U}(N)$. The tangent space to the symmetric space $\mathrm{U}(N)/\mathrm{O}(N)$ is the space of real symmetric matrices (up to a factor i). The Hamiltonians of systems with time reversal and spin rotation symmetry are of this form. This is the orthogonal class in the Wigner-Dyson classification.

When the electron spin degree is included in the description, the number of degrees of freedom is doubled and the Hamiltonian is a $2N \times 2N$ Hermitian matrix. We may decompose the Hamiltonian into 2×2 blocks c_{ij} containing matrix elements between up and down spin states and express each block in the form

$$c_{ij} = (a_{ij}^0 + ib_{ij}^0)\tau_0 + (a_{ij}^1 + ib_{ij}^1)\tau_1 + (a_{ij}^2 + ib_{ij}^2)\tau_2 + (a_{ij}^3 + ib_{ij}^3)\tau_3 \,. \tag{3}$$

Here, $\tau_0 = 1_2$ the 2×2 identity matrix, $\tau_k = i\sigma_k$ ($k = 1, 2, 3$), where the σ_k are the Pauli matrices, and a_{ij}^k, b_{ij}^k ($k = 0, 1, 2, 3$) are real numbers. Since the Hamiltonian is Hermitian, the coefficients $a_{i,j}$ and $b_{i,j}$ must satisfy the following conditions

$$a_{ij}^0 = a_{ji}^0\,,\ a_{ij}^k = -a_{ji}^k\,(k=1,2,3)\,,\ b_{ij}^0 = -b_{ji}^0\,,\ b_{ij}^k = b_{ji}^k\,(k=1,2,3)\,. \tag{4}$$

Using (3) a general Hamiltonian may be decomposed into $H = H_1 + H_2$ where H_1 is a matrix with c_{ij} of the form

$$c_{ij} = a_{ij}^0\tau_0 + a_{ij}^1\tau_1 + a_{ij}^2\tau_2 + a_{ij}^3\tau_3\,, \tag{5}$$

and H_2 is the remainder, i.e. involving the $b_{ij}^k\,(k = 0,1,2,3)$. We may define $X = iH_2$. The matrices X satisfy

$$JX + X^{\mathrm{T}}J = 0\,,\ J_{ij} = \delta_{ij}\tau_2\,, \tag{6}$$

and are the elements of a Lie algebra that is a subalgebra of $\mathrm{u}(2N)$. The corresponding Lie group is $\mathrm{Sp}(2N)$. The tangent space to the symmetric space $\mathrm{U}(2N)/\mathrm{Sp}(2N)$ is the space of matrices H_1 (up to a factor i). The Hamiltonians of systems with time reversal symmetry but where spin rotation symmetry is broken are of precisely this form. (Incidentally, this means that the Hamiltonians of such systems can be expressed as $N \times N$ matrices of quaternions, which can simplify both analytic and numerical calculations.) This is the symplectic class in the Wigner-Dyson classification.

In certain physical problems in disordered systems we encounter Hamiltonians of the form

$$H = \begin{pmatrix} 0_N & h \\ h^\dagger & 0_M \end{pmatrix}\,. \tag{7}$$

Here, 0_N denotes the $N \times N$ zero matrix, and h an $N \times M$ matrix. This describes the situation where the diagonal elements (potential energies) are vanishing and hopping is allowed only between different sublattices. Such a Hamiltonian satisfies

$$H = -\begin{pmatrix} 1_N & 0 \\ 0 & -1_M \end{pmatrix} H \begin{pmatrix} 1_N & 0 \\ 0 & -1_M \end{pmatrix}\,. \tag{8}$$

This property is called chiral symmetry [10, 11]. We may decompose a general $(N+M)\times(N+M)$ Hamiltonian into the form $H = H_1 + H_2$ where H_1 has the form (7) and H_2 is the remainder, which has the form

$$H_2 = \begin{pmatrix} h_N & 0 \\ 0 & h_M \end{pmatrix}\,. \tag{9}$$

Class	T	S	Symmetric space	Symbol
Wigner-Dyson	Yes	Yes	$\mathrm{U}(N)/\mathrm{O}(N)$	AI
Wigner-Dyson	Yes	No	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	AII
Wigner-Dyson	No	irrlevant	$\mathrm{U}(N)$	A
chiral	Yes	Yes	$\mathrm{SO}(N+M)/(\mathrm{SO}(N)\times\mathrm{SO}(M))$	BDI
chiral	Yes	No	$\mathrm{Sp}(2N+2M)/(\mathrm{Sp}(2N)\times\mathrm{Sp}(2M))$	CII
chiral	No	irrelevant	$\mathrm{U}(N+M)/(\mathrm{U}(N)\times\mathrm{U}(M))$	AIII
BdG	Yes	Yes	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	CI
BdG	Yes	No	$\mathrm{SO}(2N)/\mathrm{U}(N)$	DIII
BdG	No	Yes	$\mathrm{Sp}(2N)$	C
BdG	No	No	$\mathrm{SO}(2N)$	D

TABLE II: Ten universality classes and corresponding Lie group (symmetric spaces). The last column is the symbol for the symmetry class.

We may then define $X = iH_2$. The matrices X are the elements of a Lie algebra with corresponding Lie group $\mathrm{U}(N)\times\mathrm{U}(M)$. The tangent space of the symmetric space $\mathrm{U}(N+M)/(\mathrm{U}(N)\times\mathrm{U}(M))$ are precisely the Hamiltonians of the form (7) (up to a factor i).

In disordered superconductors, quasi-particles are described by a Bogoliubov-de Gennes (BdG) Hamiltonian of the form [12],

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^{\mathrm{T}} \end{pmatrix}, \tag{10}$$

which satisfies

$$H = -\begin{pmatrix} 0_N & 1_N \\ 1_N & 0_N \end{pmatrix} H^{\mathrm{T}} \begin{pmatrix} 0_N & 1_N \\ 1_N & 0_N \end{pmatrix} = -\tau_x H^{\mathrm{T}} \tau_x \,, \tag{11}$$

where $1_N, 0_N$ are N-dimensional unit and zero matrices, respectively. By a unitary rotation of the basis we may define $H' = g^\dagger H g\,,\ g = (1_{2N} + i\tau_x)/\sqrt{2}$ that satisfies $H' = -H'^{\mathrm{T}}$. We may then define matrices $X = iH'$. The matrices are elements of a Lie algebra $\mathrm{so}(2N)$ with the corresponding Lie group $\mathrm{SO}(2N)$.

By imposing time reversal and spin rotation symmetries, in addition to the chiral or BdG symmetries, we arrive at the ten universality classes listed in Table II.

Random Hamiltonians can be mapped to non-linear sigma models [2, 13]. Reflecting the symmetries of the Hamiltonian, the non-linear sigma models are associated with different symmetric spaces. More details can be found in review articles such as [14–16].

ANDERSON TRANSITION IN TWO DIMENSIONS

The scaling theory of localization [17] predicts that all states are localized in 2D. The argument rests on two assumptions. First, that the conductance g (in units of e^2/h) of a system of size L obeys a single parameter scaling law

$$\beta(g) = \frac{\mathrm{d}\log g(L)}{\mathrm{d}\log L} \,. \tag{12}$$

And, second, that this β-function is monotonic. For a d-dimensional system, perturbation theory in $1/g$ gives $\beta(g) \approx d - 2 - a/g$ for large g. While for small g, where $g(L) \sim \exp(-bL/\xi)$, we expect $\log(g)$. (Here, a, b are numerical factors of the order of 1.) If we interpolate the beta function between large and small g, assuming it to be monotonic, we find that $\beta(g) < 0$ for all g, and hence that the conductance always vanishes in the limit of large L.

However, this argument applies only to the Wigner-Dyson orthogonal class and, in fact, this is the only symmetry class where in 2D the states are always localized and $g(L)$ scales to zero. All the remaining universality classes listed in Table II permit the existence of delocalized states in 2D.

Wigner-Dyson symplectic class

The Wigner-Dyson symplectic class is realized in systems with spin-orbit scattering, which conserves time reversal symmetry but breaks spin rotation symmetry. One of the model Hamiltonians used in the study of this class is the so called SU(2) model [18],

$$H = \sum_{i,\sigma} \epsilon_i c^\dagger_{i\sigma} c_{i\sigma} - \sum_{(i,j)\sigma,\sigma'} R(i,j)_{\sigma,\sigma'} c^\dagger_{i\sigma} c_{j\sigma'} \,, \tag{13}$$

with i, j the site indices, σ, σ' the spin indices, (i, j) pairs of nearest neighboring sites, and $R(i, j)$ a random SU(2) matrix uniformly distributed with respect to the group invariant measure. The on-site random potential ϵ_i is uniformly distributed in the range

$$-\frac{W}{2} < \epsilon_i < \frac{W}{2} \,. \tag{14}$$

In contrast to other models, where corrections to scaling need to be taken into account [19], the SU(2) model in 2D exhibits vanishingly small corrections to scaling. This property enables high precision finite size scaling analyses of the Lyapunov exponent [3, 20]. The critical exponent ν for SU(2) model has been found to be [18, 21],

$$\nu = 2.75 \pm 0.01 \,. \tag{15}$$

At the transition, where the length scale ξ diverges, the correlation function of the local density of states $\rho(E, \vec{r})$ has a power law decay

$$\langle \rho(E, \vec{r}) \rho(E, \vec{r}\,') \rangle \sim |\vec{r} - \vec{r}\,'|^{-2\eta} \,, \tag{16}$$

while in a quasi-1D system it has an exponential decay,

$$\langle \rho(E, \vec{r}) \rho(E, \vec{r}\,') \rangle_{\rm q1D} \sim \exp(-2|\vec{r} - \vec{r}\,'|/\xi_{\rm q1D}) \,. \tag{17}$$

It follows from the assumption of conformal invariance at the critical point that these decays are related by [22, 23]

$$\frac{\xi_{\rm q1D}}{L} = \frac{1}{\pi\eta} \,, \tag{18}$$

with L the width of quasi-1D strip. In numerical simulations of the Anderson transition, the finite size scaling analysis of the Lyapunov exponent λ, which is estimated using the transfer matrix method [3, 20, 24], has proved to be the most powerful and precise approach. For the Lyapunov exponent it has been claimed [25, 26] that conformal invariance implies a different relation of the form

$$\lambda L = \pi(\alpha_0 - 2) \,, \tag{19}$$

with α_0 the position of the maximum of the multifractal spectrum $f(\alpha)$ of the distribution of the wave function (for a review, see for example [16]). There is strong numerical evidence in support of this relation [18, 27]. This suggests that the critical point in the symplectic class has conformal symmetry.

$\mathbb{Z}_2$ topological phase

In some systems in the Wigner-Dyson symplectic class, there appear edge states in the insulating phase that are stable against perturbations that preserve time reversal

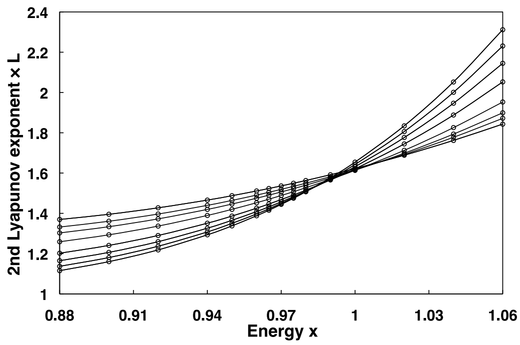

FIG. 1: The product of the second smallest positive Lyapunov exponent and the system size versus energy for the $\mathbb{Z}_2$ network model near the metal-topological insulator transition. The lines are a finite size scaling fit, with different lines corresponding to different system sizes. The critical exponent, $\nu = 2.77 \pm 0.06$ [30], is consistent with (15).

symmetry. Such edge states carry current, resulting in quantized conductance $2e^2/h$, which is observed experimentally [28]. If there are even numbers of edge states at one edge of a system, they are mixed and back scattered, resulting in vanishing conductance. On the other hand, in the case of an odd number of edge states, one edge state survives and the conductance is again $2e^2/h$. This insulating phase is called the $\mathbb{Z}_2$ topological insulator.

The critical exponent of the metal-topological insulator transition was conjectured to be the same as for the ordinary metal-insulator transition [29] and this has recently been confirmed by an analysis of the second smallest Lyapunov exponent [30] (Fig 1).

In disordered systems, the conductance is a statistical quantity and not the value but the distribution of conductance is important. (In this sense, the conductance that appears in the β-function should be regarded as a suitable average.) The distribution function, $P(g)$ should obey the scaling form [31],

$$P(g) = f(g, L/\xi)\,. \tag{20}$$

At the Anderson transition $\xi \to \infty$, the distribution becomes system size independent. This size independent distribution is called the critical conductance distribution and is denoted by $P_c(g)$. It has been demonstrated that $P_c(g)$ is universal, i.e. model independent (see, for example, [32]). However, the form of this universal distribution has been found to be different depending on whether the adjacent insulating phase is a topological insulator or not [32] (Fig. 2).

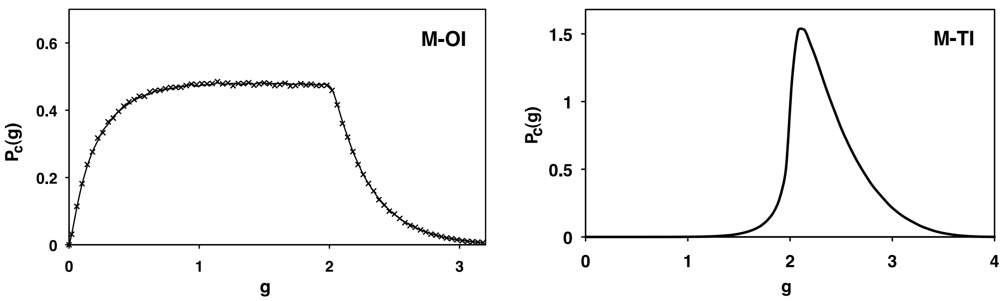

FIG. 2: Critical conductance distributions at the metal-ordinary insulator (M-OI) transition and at the metal-$\mathbb{Z}_2$ topological insulator (M-TI) transition. The distributions shown are for the $\mathbb{Z}_2$ network model (solid lines) and the SU(2) model ($\times$). See [32] for details.

DISCUSSION

The properties of the Anderson transition in 2D systems in the Wigner-Dyson symplectic class (multifractal distribution of the wave function amplitude, conformal invariance of the critical theory, universality of the conductance distribution etc.) are conjectured to apply also to other universality classes. The most important case is the unitary class where the Anderson transition is the quantum Hall transition [33]. The quantum Hall insulator is a $\mathbb{Z}$ topological insulator, where the current is carried by an integer number of edge states. Confirmation of conformal invariance, however, is still pending. The central difficulty is that corrections to scaling vanish only very gradually with increasing system size [34]. Further efforts, both numerical as well as analytic, will be needed to understand the quantum Hall transition quantitatively.

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[1] P. Anderson, Phys. Rev., **109**, 1492 (1958).

[2] K. Efetov, *Supersymmetry in disorder and chaos* (Cambridge Univ. Press, 1997).

[3] B. Kramer, A. MacKinnon, Rep. Prog. Phys., **56**, 1469 (1993).

[4] K. Itoh *et al.*, J. Phys. Soc. Jpn., **73**, 173 (2004).

[5] S. John, Phys. Today, **44**, 32 (1991).

[6] S. Faez, A. Strybulevych, J.H. Page, Ad Lagendijk, B.A. van Tiggelen, Phys. Rev. Lett., **103**, 155703 (2009).

[7] J. Chabe *et al.*, Phys. Rev. Lett., **101**, 255702 (2008).

[8] E.P. Wigner, Proc. Cambridge Philos. Soc., **47**, 790 (1951).

[9] F.J. Dyson, J. Math. Phys., **3**, 140,157,166 (1962).

[10] R. Gade, F. Wegner, Nuclear Physics B, **360**, 213 (1991).

[11] R. Gade, Nuclear Physics B, **398**, 499 (1993).

[12] A. Altland, M.R. Zirnbauer, Phys. Rev. B, **55**, 1142 (1997).

[13] S. Hikami, Phys. Rev. B, **24**, 2671 (1981).

[14] M.R. Zirnbauer, arXiv:math-ph/9808012.

[15] B. Kramer, T. Ohtsuki, S. Kettemann, Phys. Rep., **417**, 211 (2005).

[16] F. Evers, A.D. Mirlin, Rev. Mod. Phys., **80**, 1355 (2008).

[17] E. Abrahams *et al.*, Phys. Rev. Lett., **42**, 673 (1979).

[18] Y. Asada, K. Slevin, T. Ohtsuki, Phys. Rev. Lett., **89**, 256601 (2002).

[19] K. Slevin, T. Ohtsuki, Phys. Rev. Lett., **82**, 382 (1999).

[20] A. MacKinnon, B. Kramer, Z. Phys., **B53**, 1 (1983).

[21] Y. Asada, K. Slevin, T. Ohtsuki, Phys. Rev. B, **70**, 035115 (2004).

[22] J.L. Cardy, J. Phys. A, **17**, L385 (1984).

[23] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, 1996).

[24] J-L. Pichard, G. Sarma, J. Phys. C, **14**, L127 (1981).

[25] A. Dohmen, P. Freche, M. Janssen, Phys. Rev. Lett., **76**, 4207 (1996).

[26] M. Janssen, Phys. Rep., **295**, 1 (1998).

[27] H. Obuse, R. Subramaniam, A. Furusaki, I. A. Gruzberg, A.W.W. Ludwig, Phys. Rev. Lett., **98**, 156802 (2007).

[28] M. König *et al.*, Science, **318**, 766 (2007).

[29] H. Obuse, A. Furusaki, S. Ryu, C. Mudry, Phys. Rev. B, **76**, 075301 (2007).

[30] K. Kobayashi, T. Ohtsuki, K. Slevin, unpublished.

[31] B. Shapiro, Phys. Rev. Lett., **65**, 1510 (1990).

[32] K. Kobayashi, T. Ohtsuki, H. Obuse, K. Slevin, Phys. Rev. B, **82**, 165301 (2010).

[33] M. P. K. von Klitzing, G. Dorda, Phys. Rev. Lett., **45**, 494 (1980).

[34] K. Slevin, T. Ohtsuki, Phys. Rev. B, **80**, 041304(R) (2009).